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In the comoving frame, with the metric (5) we obtain

$$h^{\alpha}{}_{;\alpha} = 0. \tag{17}$$

We assume further the existence of a space-like Killing vector field proportional to h_{μ}. Then if ρ, ϵ and $|\boldsymbol{h}|^2$ are constant we can integrate equation (16) and obtain the barometric formula

$$p = (\rho c^2 + \rho\epsilon + \mu c^2 |\boldsymbol{h}|^2)\left(\frac{\xi_s}{\xi} - 1\right)$$

where ξ_s is the absolute value of the Killing vector at the surface of the considered domain of space–time.

We note that the magnetic field contributes to the fluid pressure twice: through the proper energy density $\frac{1}{2}\mu c^2|\boldsymbol{h}|^2$ and through the magnetic pressure $\frac{1}{2}\mu c^2|\boldsymbol{h}|^2$. This conclusion is in full accordance with the fundamental concepts of relativistic magnetohydrodynamics.

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An exact solution of the multiatom, multimode model Hamiltonian of quantum optics

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Abstract. Exact expressions for the eigenvectors and eigenvalues of a Hamiltonian of great importance in quantum optics are derived.

The Hamiltonian we shall consider is

$$H = \sum_{k=1}^{N} a_k{}^{\dagger} a_k w_k + R_3 w_0 + \sum_{k=1}^{N} (g a_k R_+ + g^* a_k{}^{\dagger} R_-) \tag{1}$$

where $R_\alpha = \Sigma_{i=1} \frac{1}{2}\sigma_{i\alpha}$, the $\sigma_{i\alpha}$ being the Pauli spin matrices for the ith atom, and $a_k{}^{\dagger}$ is the creation operator for the kth mode of the electromagnetic field. This Hamiltonian, which describes a system of N electromagnetic modes interacting with M two level atoms, is of considerable interest in quantum optics. Arecchi *et al* (1969) give a discussion of the Hamiltonian and its significance (see also Pike and Swain 1970). It is of interest, too, in the theory of paramagnetic ions in crystals.

The Hamiltonian was solved exactly for the case of one atom and one mode by Jaynes and Cummings (1963). Tavis and Cummings (1968) obtained an exact solution to the multiatom, single mode Hamiltonian for the particular case of the frequency of the mode being equal to w_0. Mallory (1969) (see also Walls and Barakat 1970) was able to remove the latter restriction. Here we present an exact solution of the Hamiltonian (1) without making any simplifying assumptions.

The essential step is to notice that if we decompose the Hamiltonian as follows:

$$H = C+Q \tag{2}$$

where

$$C = \left(\sum_k a_k{}^\dagger a_k + R_3\right) w_0 \tag{3}$$

and

$$Q = \sum_k (a_k{}^\dagger a_k w_{k0} + g a_k R_+ + g^* a_k{}^\dagger R_-) \qquad w_{k0} = w_k - w_0 \tag{4}$$

then C and Q commute with each other, and consequently with H. (Of course they also commute with $R^2 = \frac{1}{2}(R_+R_- + R_-R_+) + R_3{}^2$. Any eigenvector of H must also be an eigenvector of R^2.) Thus if we can find simultaneous eigenvectors of C and Q (and R^2) they will be eigenvectors of H. Now an eigenvector of C with eigenvalue $cw_0 = (\Sigma n_k + m)w_0$ is $|n_1 n_2 \ldots n_N\rangle |r, m\rangle$ where

$$a_j{}^\dagger a_j |n_1 n_2 \ldots n_j \ldots n_N\rangle = n_j |n_1 n_2 \ldots n_j \ldots n_N\rangle \qquad n_j = 0, 1, 2, \ldots \tag{5}$$

$$R^2 |r, m\rangle = r(r+1)|r, m\rangle \qquad r = \tfrac{1}{2}M, \tfrac{1}{2}M - 1 \ldots \tag{6}$$

and

$$R_3 |r, m\rangle = m|r, m\rangle \qquad m = +r, r-1, \ldots, -r. \tag{7}$$

Clearly we must have $c \geqslant -r$, because $(\Sigma n_k + m) \geqslant -r$. A more general eigenstate is obtained by taking a linear combination of all those eigenstates belonging to the same value of cw_0, that is, $C|r, c\rangle = cw_0|r, c\rangle$ where

$$|r, c\rangle = \sum_{m=-r}^{+r} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} A^{(r,c)}_{n_1 n_2 \ldots n_N} |n_1 n_2 \ldots n_N\rangle |r, m\rangle \times \delta(c - \Sigma n_k - m). \tag{8}$$

Performing the sum over m gives

$$|r, c\rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} A^{(r,c)}_{n_1 n_2 \ldots n_N} |n_1 n_2 \ldots n_N\rangle |r, c - \Sigma n_k\rangle \times \Delta(c + r \geqslant \Sigma n_k \geqslant c - r, 0) \tag{9}$$

where $\Delta(c+r \geqslant \Sigma n_k \geqslant c-r, 0) = 1$ if $c+r \geqslant \Sigma n_k \geqslant (c-r$ if $c \geqslant r$ and 0 if $c < r)$ and is zero otherwise. (9) is an eigenstate of C. If we can restrict the terms in the sum so that (9) is also an eigenstate of Q, then we have a simultaneous eigenstate of C and Q, which is consequently an eigenstate of H.

Operating with Q on (9), one obtains after some manipulation

$$Q|r,c\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}\sum_{k=1}^{N}\{A^{(r,c)}_{n_1n_2\ldots n_N}n_kw_{k0}+g(n_k+1)^{1/2} \times A^{(r,c)}_{n_1\ldots(n_k+1)\ldots n_N}f(c-\Sigma n_k-1)+g^*\sqrt{n_k}f(c-\Sigma n_k) \times A^{(r,c)}_{n_1\ldots(n_k-1)\ldots n_N}\}|n_1n_2\ldots n_N\rangle \times|r,c-\Sigma n_k\rangle\Delta(c+r\geqslant\Sigma n_k\geqslant c-r,0) \quad (10)$$

where $f(m)$ is defined by

$$f(m) = \{r(r+1)-m(m+1)\}^{1/2}. \quad (11)$$

We note that

$$R_+|r,m\rangle = f(m)|r,m+1\rangle$$
$$R_-|r,m\rangle = f(m-1)|r,m-1\rangle.$$

In obtaining (10) we have made use of the properties $f(r) = f(-r-1) = 0$.

It follows from (10) that we can make (9) an eigenvector of Q belonging to the value q if we make the A satisfy the system of equations

$$\left(\sum_k n_kw_{k0}-q\right)A^{(r,c)}_{n_1n_2\ldots n_N}+\sum_k\{g(n_k+1)^{1/2}f(c-\Sigma n_k-1) \times A^{(r,c)}_{n_1n_2\ldots(n_k+1)\ldots n_N}+g^*\sqrt{n_k}f(c-\Sigma n_k)A^{(r,c)}_{n_1n_2\ldots(n_k-1)\ldots n_N}\}=0 \quad (12)$$

for all positive integer values of $n_1, n_2, \ldots n_N$ such that $c+r\geqslant\Sigma n_k\geqslant c-r, 0$. If we specialize to one mode and put $w_{k0} = 0$, this expression reduces to the equation (2.8) of Tavis and Cummings.

The equations (12) are a set of homogeneous linear equations. The condition for consistent solutions leads to a polynomial equation for the eigenvalues q. For an allowed value of q the A can in principle be determined to within an arbitrary multiplicative factor. This factor is fixed (apart from a phase factor) by the condition that the eigenvector be normalized. Let us denote by $|r, c, q\rangle$ a vector of the form (9) whose coefficients satisfy (12). We then have

$$Q|r,c,q\rangle = q|r,c,q\rangle \qquad H|r,c,q\rangle = (cw_0+q)|r,c,q\rangle. \quad (13)$$

Thus $|r, c, q\rangle$ is an eigenvector of H having the eigenvalue (cw_0+q).

It is a formidable task to solve analytically the set of equations (12) in the general case, and numerical methods are normally required. However, solutions for small values of r and c can be easily obtained. For example, consider a system of two atoms and two modes. Let us take $r = 1$. If we take $c = -1$, then it is easily seen that the only values of n_1 and n_2 to be considered in (12) are $n_1 = n_2 = 0$, and thus the only value of q allowed is $q = 0$. Thus the eigenstate $|1, -1, 0\rangle = |0, 0\rangle|1, -1\rangle$ which is identical with the ground state of the unperturbed system corresponding to both atoms being in their ground state with no photons present. Now take $c = 0$. (12) gives us the three equations

$$-qA_{00}{}^{(1,0)}+\sqrt{2}g(A_{10}{}^{(1,0)}+A_{01}{}^{(1,0)}) = 0 \quad (14)$$

$$(-q+w_{20})A_{01}{}^{(1,0)}+\sqrt{2}g^*A_{00}{}^{(1,0)} = 0 \quad (15)$$

$$(-q+w_{10})A_{10}{}^{(1,0)}+\sqrt{2}g^*A_{00}{}^{(1,0)} = 0 \quad (16)$$

which can be solved to give

$$A_{01}{}^{(1,0)} = \frac{\sqrt{2}g^* A_{00}}{q-w_{20}} \qquad A_{10}{}^{(1,0)} = \frac{\sqrt{2}g^* A_{00}}{q-w_{10}} \tag{17}$$

and q satisfies

$$q(q-w_{10})(q-w_{20})-2|g|^2(2q-w_{20}-w_{10}) = 0. \tag{18}$$

There are thus three eigenstates in this case, labelled by the three values of q. (18) becomes particularly simple if we choose $w_1 = w_0-\delta$, $w_2 = w_0+\delta$. Then

$$q = 0 \quad \text{or} \quad q = \pm a \qquad a = (\delta^2+4|g|^2)^{1/2}$$

and from (9) and (17) the corresponding eigenstates are

$$|1,0,0\rangle = A_{00}{}^{(1,0)}(0)\left(|0,0\rangle|1,0\rangle+\frac{\sqrt{2}g^*}{\delta}(|1,0\rangle|1,-1\rangle-|0,1\rangle|1,-1\rangle)\right)$$

$$|1,0,+a\rangle = A_{00}{}^{(1,0)}(a)\left\{|0,0\rangle|1,0\rangle + \sqrt{2}g^*\left(\frac{|1,0\rangle|1,-1\rangle}{a+\delta}+\frac{|0,1\rangle|1,-1\rangle}{a-\delta}\right)\right\}$$

$$|1,0,-a\rangle = A_{00}{}^{(1,0)}(-a)\left\{|0,0\rangle|1,0\rangle - \sqrt{2}g^*\left(\frac{|1,0\rangle|1,-1\rangle}{a-\delta}+\frac{|0,1\rangle|1,-1\rangle}{a+\delta}\right)\right\}.$$

The $A_{00}{}^{(1,0)}(q)$ are obtained using the normalization condition.

If one took the case $r = 1$, $c = 2$, one would have a quintic to solve for q, giving five eigenvalues of H with $r = 1$, $c = 2$, and so on. The algebra increases rapidly in complexity as r and c increase in value. The results of a numerical study of the system (12) will be published at a later date.

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